ON THE SPECTRUM OF INFINITE DIMENSIONAL RANDOM PRODUCTS OF COMPACT OPERATORS

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Abstract. We consider an infinite dimensional separable Hilbert space and its family of compact integrable cocycles over a dynamical system f. Assuming that f acts in a compact Hausdorff space X and preserves a Borel regular ergodic measure which is positive on non-empty open sets, we conclude that there is a residual subset of cocycles within which, for almost every x, either the Oseledets-Ruelle's decomposition along the orbit of x is dominated or has a trivial spectrum.

MSC 2000: primary 37H15, 37D08; secondary 47B80.
keywords: Random operators; dominated splitting; multiplicative ergodic theorem; Lyapunov exponents.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{C}(\mathcal{H})$ the set of linear compact operators acting in $\mathcal{H}$ with the uniform norm given by

$$\|\mathcal{T}\| = \sup_{v\neq 0}\frac{\|\mathcal{T}(v)\|}{\|v\|}.$$

Consider a homeomorphism $f : X \to X$ of a compact Hausdorff space X and μ an f-invariant Borel regular measure that is positive on non-empty open subsets. Given a family $(A(x))_{x\in X}$ of operators in $\mathcal{C}(\mathcal{H})$ and a continuous vector bundle $\pi : X\times\mathcal{H} \to X$, we define the *associated cocycle over* f by

$$\begin{array}{rccc} F(A): & X\times\mathcal{H} & \longrightarrow & X\times\mathcal{H} \\ & (x,v) & \longmapsto & (f(x), A(x)\cdot v). \end{array}$$

The map F satisfies the equality $\pi\circ F = f\circ\pi$ and, for all $x\in X$, $F_x(A) : \mathcal{H}\to\mathcal{H}$ is linear on the fiber $\mathcal{H} := \pi^{-1}(\{x\})$. For simplicity of notation we call A a *cocycle*.

A *random product* of a cocycle $A : X \to \mathcal{C}(\mathcal{H})$ *associated to the map* f is the sequence, indexed by $x \in X$, of linear maps of $\mathcal{H}$ defined, for

Date: September 21, 2007.

each $n \in \mathbb{N}_0$, by $A^0(x) = Id$ and

$$A^n(x) = A(f^{n-1}(x)) \circ ... \circ A(f(x)) \circ A(x).$$

In this paper we are interested in the asymptotic properties of random products, that is, the limit of the spectra, as n goes to ∞, of the sequence $(A^n(x))_{n\in\mathbb{N}}$, for most points x. In general it is not guaranteed, not even in a relevant subset of X, the convergence of the sequence of operators $(A^n(x))_{n\in\mathbb{N}}$ or of their spectra. But under the hypothesis that A is integrable, that is,

$$\int_X \log^+ \|A(x)\| \, d\mu(x) < \infty,$$

where $\log^+(y) = \max\{0, \log(y)\}$, the theorem of Ruelle ([8]) offers, for μ-almost every point $x \in X$, a nice description of a complete set of Lyapunov exponents and associated A-invariant directions. The aim of this work is to identify generic properties of these exponents and corresponding decomposition.

The approach in Mate's work ([5]), where it is assumed that A is a bounded operator, f is the shift of N symbols and, for every x, the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, suggests that the null cocycle has a main role in this context: we may split the Hilbert space into a direct sum of two subspaces, one that aggregates all the fixed directions and the other corresponding to the eigenvalue zero (that is, the Lyapunov exponent $-\infty$). Among compact cocycles this scenario should be improved. In fact, for these operators the unique point of accumulation of the spectrum is 0 and therefore the component of the spectrum that may lie on the unit circle (inducing non-hyperbolicity) is finite dimensional; besides if the spectrum is trivial (reduced to one point), then the compact operator has to be the null one. Nevertheless the success of Mate's result, which does not depend on perturbations, is strongly based upon the hypothesis that, for every shift orbit, the sequence of operators $(A^n(x))_{n\in\mathbb{N}}$ converges. Without this assumption, the best we can expect is an approximate result stating that, generically, either the above Mate's decomposition reduces to the null part or is, in some sense, hyperbolic.

The main difficulty, due to the infinite dimensional environment, is precisely to cancel the spectrum by a small perturbation of the original system. In the context of families of finite dimensional linear invertible cocycles, Bochi and Viana ([3]) managed to prove that, by a C^0-small perturbation, we may reach a cocycle exhibiting, for almost every point, uniform hyperbolicity in a finite projective space or else a one-point spectrum Oseledets' decomposition. By *hyperbolicity* the authors mean the existence, for μ-almost every $x \in X$, of an $A(x)$-invariant decomposition of the fiber $\mathcal{H}$ into a direct sum of two invariant subspaces $E^1_x \oplus E^2_x$ which varies continuously with the point x and enhances a stronger contraction, or a weaker expansion, by A along the first one.

In our setting we could apply directly this result to a C^0-approximation of A with finite rank (see [6]); however this straight application would endorse a meagre result: it only gives a C^0-dense panorama, instead of the aimed C^0-residual one; besides, in the case a dominated splitting prevails, this would be a decomposition of just a finite dimensional subspace of $\mathcal{H}$.

Essentially all we need is a strategy to perturb and therefore to produce a residual dichotomy; this has to be done without leaving the world $C^0_I(X, \mathcal{C}(\mathcal{H}))$ of continuous compact integrable cocycles and keeping control on the possibly infinite amount of Lyapunov exponents, most of which may be equal to $-\infty$. Two key ingredients in the argument of [3] can be adapted to our infinite dimensional context: the upper semi-continuity of a map that measures how the sum of Lyapunov exponents behaves; and the extension and continuity of a dominated splitting. Concerning the first one, we had to accept that now this map has infinite components and, due to the presence of the Lyapunov exponent $-\infty$, may take values on the extended real set, which may prevent integrability. This difficulty is the reason for assuming that μ is ergodic and positive on non-empty open sets. But this is the main difference, in the large its intervention is the same as in [3]. The second one is harder to deal with because $\mathcal{H}$ is infinite dimensional and A, being compact, is not invertible - and it may even happen that $\inf \{\|A(x)\| : x \in X\} = 0$. The notion of dominated splitting must then be reformulated and applied to Oseledets-Ruelle's splittings where the stronger space is associated to the first $k \in \mathbb{N}$ finite Lyapunov exponents (whose sum of multiplicities gives the *index* of the splitting) and the weaker subspace corresponds to the remaining ones: this way the first subspace is finite dimensional and there the restriction of A is invertible. Let us see how we proceed from here.

As $C^0_I(X, \mathcal{C}(\mathcal{H}))$ is a Baire space and each p^{th}-component of the entropy map is upper semi-continuous (see Section 2.5), each has a residual subset $\mathcal{R}_p$ of continuity points. The set $\cap \mathcal{R}_p$ is also residual and its elements are points of continuity of all of these map-components. We take one of them, say A, and apply to it Ruelle's theorem (see Section 2.2). As μ is ergodic, the Lyapunov exponents of $A(x)$ and corresponding multiplicities are constant for μ-almost every x. Besides, as μ is positive on non-empty open subsets, the properties that are valid μ - almost everywhere are also dense.

By compactness of the operators, if the Lyapunov exponents of A are all equal, then they must be $-\infty$ and so the limit operator given by Ruelle's theorem is identically null. Assume now that the Lyapunov exponents of A are not all equal. The space X can then be sliced into measurable strata within each of which the Oseledets-Ruelle's decomposition induces a direct sum $\mathcal{H} = E_1 \oplus E_2$ where the dimension of E_1 is

constant, E_1 is associated to some finite number of the first finite Lyapunov exponents, and the splitting is dominated. If the union of these slices has full measure, the proof is complete. Otherwise, we can find a subset with positive measure where neither the Oseledets-Ruelle's splitting is dominated nor the Lyapunov exponents are all equal. This allows us to diminish drastically, by a small global perturbation, the value of one of the components of the entropy map, contradicting its continuity at A. Accordingly we establish that:

Theorem 1.1. *There exists a C^0-residual subset $\mathcal{R}$ of the set of integrable compact cocycles $C^0_I(X, \mathcal{C}(\mathcal{H}))$ such that, for $A \in \mathcal{R}$ and μ-almost every $x \in X$, either the limit $\lim_{n\to\infty} (A(x)^{*n} A(x)^n)^{\frac{1}{2n}}$ is the null operator or the Oseledets-Ruelle's splitting of A along the orbit of x is dominated.*

2. Preliminary results

2.1. Completeness.

Lemma 2.1. *$C^0_I(X, \mathcal{C}(\mathcal{H}))$ is a Baire space.*

Proof. Since $\mathcal{H}$ is complete, the space $\mathcal{C}(\mathcal{H})$, with the uniform norm, is also complete (see [6]). The space $C^0(X, \mathcal{C}(\mathcal{H}))$ is endowed with the norm defined by

$$\|A\| = \max_{x \in X} \|A(x)\|$$

and this way it is complete: if $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0(X, \mathcal{C}(\mathcal{H}))$ then, for each $x \in X$, the sequence $(A_n(x))_{n\in\mathbb{N}}$ has the Cauchy property and therefore converges in $\mathcal{C}(\mathcal{H})$. This defines a limit of $(A_n)_{n\in\mathbb{N}}$ in $C^0(X, \mathcal{C}(\mathcal{H}))$.

Consider now a Cauchy sequence, say $(B_n)_{n\in\mathbb{N}}$, of elements of the subspace $C^0_I(X, \mathcal{C}(\mathcal{H}))$, that is, continuous compact cocycles such that, for all $n \in \mathbb{N}$, $\int_X \log^+ \|B_n(x)\| \, d\mu(x) < \infty$. Then:

- $(B_n)_{n\in\mathbb{N}}$ converges to some $B \in C^0(X, \mathcal{C}(\mathcal{H}))$.
- As X is compact and B is continuous, there exists $M > 0$ such that, for all $x \in X$, we have $\|B(x)\| \leq M$; and so $0 \leq \log^+ \|B(x)\| \leq \log(M)$.
- As $(\|B_n\|)_n$ converge uniformly to $\|B\|$, the same holds for the sequence of μ - integrable maps $(\log^+ \|B_n\|)_n$, and therefore $\log^+(\|B\|)$ is μ - integrable.
- Besides $0 \leq \int_X \log^+ \|B(x)\| \, d\mu(x) \leq \log(M) < \infty$.

□

2.2. The multiplicative ergodic theorem.

The following result gives a spectral decomposition for the limit of random products of compact cocycles under the previously defined integrability condition.

Theorem 2.2. *(Ruelle [7]) Let $f : X \to X$ be a homeomorphism and μ any f-invariant Borel probability. If A belongs to $C^0_I(X, \mathcal{C}(\mathcal{H}))$, then, for μ-a.e $x \in X$, we have the following properties:*

(a) *The limit $\lim_{n\to\infty} (A(x)^{*n} A(x)^n)^{\frac{1}{2n}}$ exists and is a compact operator $\mathcal{L}(x)$, where A^* denotes the dual operator of A.*

(b) *Let $e^{\lambda_1(x)} > e^{\lambda_2(x)} > \ldots$ be the nonzero eigenvalues of $\mathcal{L}(x)$ and $U_1(x)$, $U_2(x)$, ... the associated eigenspaces whose dimensions are denoted by $n_i(x)$. The sequence of real functions $\lambda_i(x)$, called **Lyapunov exponents** of A, where $1 \leq i(x) \leq j(x)$ and $j(x) \in \mathbb{N} \cup \{\infty\}$ verifies:*

(b.1) *The functions $\lambda_i(x)$, $i(x)$, $j(x)$ and $n_i(x)$ are f-invariant and depend in a measurable way on x.*

(b.2) *Let $V_i(x)$ be the orthogonal complement of $U_1(x) \oplus U_2(x) \oplus \ldots \oplus U_{i-1}(x)$ for $i < j(x) + 1$ and $V_{j(x)+1}(x) = Ker(\mathcal{L}(x))$. Then:*

(i) $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)u\| = \lambda_i(x)$ *if* $u \in V_i(x) \setminus V_{i+1}(x)$ *and* $i < j(x) + 1$*;*

(ii) $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)u\| = -\infty$ *if* $u \in V_{j(x)+1}(x)$*.*

Notice that, as μ is ergodic, the maps $i(x)$, $j(x)$, $n_i(x)$ and $\lambda_i(x)$ are constant μ-almost everywhere. Besides, as $\mathcal{L}(x)$ is a compact operator, if its eigenvalues are all equal, then they must be all zero, that is, the Lyapunov exponents of A at x are all equal to $-\infty$.

In the sequel we will denote by $\mathcal{O}(A)$ the full measure set of points given by this theorem. Since μ is positive on non-empty open subsets, $\mathcal{O}(A)$ is dense in X.

2.3. **Dimension.** The infinite dimension of $\mathcal{H}$ brings additional trouble while dealing with Oseledets' decompositions because in the sequel we will need one of them with finite codimension. This is the aim of next lemma.

Lemma 2.3. *Let A be an integrable compact operator and $\lambda_i(x)$, $U_i(x)$ as in Ruelle's theorem. If $\lambda_i(x) \neq -\infty$, then $U_i(x)$ has finite dimension.*

Proof. The numbers $e^{\lambda_1(x)} > e^{\lambda_2(x)} > \ldots$, where $\lambda_k(x)$ is different from $-\infty$, are the nonzero eigenvalues of the compact operator $\mathcal{L}(x)$, and $U_1(x)$, $U_2(x)$, ... the associated eigenspaces. By compactness of $\mathcal{L}(x)$, theses spaces have finite dimensions (see [6]). □

2.4. **Dominated splittings.** Given f and A as above and an f-invariant set $\mathcal{K}$, we say that a splitting $E_1(x) \oplus E_2(x) = \mathcal{H}$ is *ℓ-dominated* in $\mathcal{K}$ if $A(E_i(x)) \subset E_i(f(x))$ for every $x \in \mathcal{K}$, the dimension of $E_i(x)$ is constant in $\mathcal{K}$ for $i = 1, 2$, and there are $\theta_{\mathcal{K}} > 0$ and $\ell \in \mathbb{N}$ such that, for every $x \in \mathcal{K}$ and any pair of unit vectors $u \in E_2(x)$ and $v \in E_1(x)$, one has

$$\|A(x)(v)\| \geq \theta_{\mathcal{K}}$$

$$\frac{\|A^\ell(x)u\|}{\|A^\ell(x)v\|} \le \frac{1}{2}.$$

This definition corresponds to hyperbolicity in an infinite dimensional projective space; we will denote it by $E_1 \succ_\ell E_2$.

The splittings we are interested in are the ones corresponding to Lyapunov subspaces given by Ruelle's theorem. In this setting:

Definition 2.1. *Given an f-invariant set $\mathcal{K}$ contained in $\mathcal{O}(A)$, the Oseledets-Ruelle's decomposition is ℓ-dominated in $\mathcal{K}$ if we may detach in it a direct sum of two subspaces, say $E_1(x) \oplus E_2(x) = \mathcal{H}$, such that $E_1(x)$ is associated to a finite number of the first Lyapunov exponents, say λ_1, λ_2, ... λ_k, the subspace $E_2(x)$ corresponds to the remaining ones and $E_1 \succ_\ell E_2$.*

The classical concept of domination in the finite dimensional setting is stronger than this one, requiring a comparison of the strength of each Oseledets' subspace with the next one. Due to the possible presence of $-\infty$ in the set of Lyapunov exponents, this is in general unattainable in our context, unless this exponent does not turn up.

Besides, for future use of the ℓ-domination, we require that the norm of A in $\mathcal{K}$ is bounded away from zero. In fact, among finite dimensional automorphisms, *domination* implies that the angle between any two subbundles of the dominated splitting is uniformly bounded away from zero, a very useful property while proving that the dominated splitting extends continuously. Due to the lack of compactness of $\mathcal{O}(A)$ and the fact that we are dealing with a family $(A(x))_x$ of compact operators acting on an infinite dimensional space - so $A(x)$ is not invertible and its norm may not be uniformly bounded away from zero - we cannot expect such a strong statement in our setting, unless we relate, as we have done in the definition, domination with non-zero norms.

The statement of next lemma ensures that we may check if x in $\mathcal{O}(A)$ has a dominated Oseledets-Ruelle's decomposition $\mathcal{H} = E_1(x) \oplus E_2(x)$, where $E_1(x)$ is the Lyapunov subspace associated to the first k finite Lyapunov exponents $\lambda_1 > \lambda_2 > ... > \lambda_k > -\infty$ and $E_2(x)$ corresponds to the remaining ones. In what follows we will address always to this specific Oseledets-Ruelle's splitting.

Lemma 2.4. *Let A be an integrable compact cocycle acting on an infinite dimensional Hilbert space $\mathcal{H}$. Consider x in $\mathcal{O}(A)$, $\lambda_1 > \lambda_2 > ... > \lambda_k$ the first k Lyapunov exponents and $E_1(x) = U_1(x) \oplus U_2(x) \oplus ... \oplus U_k(x)$ the corresponding subspace. If $\lambda_k > -\infty$, then the restriction of the operator $A(x) : E_1(x) \to E_1(f(x))$ is invertible and $A^{-1}(f(x))$ is compact.*

Proof. Let us first check that this restriction of $A(x)$ is injective. Consider $v \neq 0$ in $E_1(x)$. Then, by Ruelle's theorem, one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \geq \lambda_k > -\infty,$$

so $A(x)v$ cannot be zero. Now, by Lemma 2.3 and since μ is ergodic, the dimension of $E_1(x)$ is finite and constant in $\mathcal{O}(A)$. Therefore the map $A(x) : E_1(x) \to E_1(f(x))$ is an injective linear function between spaces of equal finite dimension, and so it is surjective. The compactness of the inverse of A, given at each point by a finite dimensional matrix, now follows. □

We need now to verify that domination is easily inherited by neighbors.

Proposition 2.5. *If the Oseledets-Ruelle's splitting $E_1(x) \oplus E_2(x) = \mathcal{H}$ is ℓ-dominated over an invariant set $\mathcal{K} \subset \mathcal{O}(A)$, it may be extended continuously to an ℓ-dominated splitting over the closure of $\mathcal{K}$.*

Proof. To extend E_1 we will take advantage from the fact that, for each $z \in \mathcal{K}$, the subspace $E_1(z)$ is an Oseledets-Ruelle's space associated to a finite number of the first finite Lyapunov exponents; and, moreover, that A is compact. The definition of E_2 at the closure is suggested by our need to extend the relation $\succ_\ell$, which is easier if the vectors at the extension are just accumulation points of sequences of vectors from where the ℓ-domination holds.

Lemma 2.6. *The angle between $E_1(z)$ and $E_2(z)$, where z belongs to $\mathcal{K}$, is uniformly bounded away from zero (say bigger than a constant $\gamma \in]0, \frac{\pi}{2}]$).*

Proof. Assume that there are sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{K}$, $(u_n)_{n\in\mathbb{N}}$ in $E_2(x_n)$ and $(v_n)_{n\in\mathbb{N}}$ in $E_1(x_n)$ such that, for all n, we have $\|u_n\| = \|v_n\| = 1$ and $u_n - v_n$ converges to 0. As A is continuous, if n is large enough then the norm $\|A^\ell(x_n)(u_n) - A^\ell(x_n)(v_n)\|$ is arbitrarily small; moreover

$$\|A^\ell(x_n)(u_n) - A^\ell(x_n)(v_n)\| \geq \|A^\ell(x_n)(u_n)\| - \|A^\ell(x_n)(v_n)\|$$

and the last difference is equal to

$$\|A^\ell(x_n)(v_n)\| \left(\frac{\|A^\ell(x_n)(u_n)\|}{\|A^\ell(x_n)(v_n)\|} - 1 \right).$$

As there is $\theta_\mathcal{K}$, independent of x_n and v_n, such that $\|A(x_n)(v_n)\| \geq \theta_\mathcal{K}$, we have $\|A^\ell(x_n)(v_n)\| \geq \theta_\mathcal{K}^\ell$ and so

$$\frac{\|A^\ell(x_n)(u_n)\|}{\|A^\ell(x_n)(v_n)\|} - 1 \approx 0.$$

But this contradicts the fact that $E_1(x_n) \succ_\ell E_2(x_n)$ for all n. □

Finally consider a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{K}$ converging to $x \in X$ and suppose that along $(x_n)_{n\in\mathbb{N}}$ we may find an ℓ-dominated splitting $E_{1,n}\oplus E_{2,n}$ made of Oseledets-Ruelle's subspaces as mentioned. Recall that $E_{1,n}$ has dimension p for all n and corresponds to the Lyapunov exponents $\lambda_1 > \lambda_2 > ... > \lambda_k > -\infty$ and that we must extend this dominated splitting to x.

Take, for each n, a unitary basis $v_{1,n}, ..., v_{p,n}$ of $E_{1,n}$. As proved above, in Lemma 2.4, since $\lambda_k > -\infty$, the restriction of the operator $A(x) : E_1(x) \to E_1(f(x))$ is invertible and $A^{-1}(f(x))$ is compact. Therefore, for each $i = 1, ..., p$, the sequence $(A(x_n)^{-1}v_{i,n})_{n\in\mathbb{N}}$ has a subsequence convergent to $h_i \in \mathcal{H}$. Apply now the operator A to obtain p vectors w_i in the fiber at x.

Claim: Each $w_i \neq 0$

In fact, by continuity of the operator A, w_i is the limit of a subsequence of $(v_{i,n})_{n\in\mathbb{N}}$ and these vectors satisfy the condition

$$\lim_{m\to\infty} \frac{1}{m} \log \|A^m(x_n)v_{i,n}\| \geq \lambda_k > -\infty$$

which implies that, if m is big enough, $\|A^m(x)w_i\| \geq \exp(m \times \lambda_k)$; this prevents w_i from being the vector zero.

We now define $E_1(x)$ as the space spanned by $w_1, ..., w_p$ and $E_2(x)$ as the set of accumulation points, when n goes to ∞, of all sequences of vectors in $E_{2,n}(x_n)$, where $(x_n)_{n\in\mathbb{N}}$ is any sequence converging to x. Notice that, as A is continuous and the spaces $E_{i,n}(x_n)$ are determined by Lyapunov exponents, we have $A(E_i(x)) \subset E_i(f(x))$.

The vectors w_i could be linearly dependent, so the dimension of $E_1(x)$ might be less than p. However, as verified above, the angle between $E_1(z)$ and $E_2(z)$, where z belongs to $\mathcal{K}$, is uniformly bounded away from zero; this prevents directions of $E_1(z)$ from mixing with those of $E_2(z)$, the way E_1 had to loose dimension. Therefore the dimension of E_1 is constant and equal to p in the closure of $\mathcal{K}$. And, as we will verify, forms with E_2 a direct sum in $\mathcal{H}$ which is a dominated splitting.

Lemma 2.7.
(a) $E_2(x)$ *is a subspace of* $\mathcal{H}$.
(b) $\mathcal{H} = E_1(x) \oplus E_2(x)$.

Proof. (a) The vector zero is in $E_2(x)$ as limit of the null sequence of vectors of the subspaces $E_{2,n}(x_n)$. Consider now a scalar η and two non-zero vectors u_0 and u_1 of $E_2(x)$. By definition of $E_2(x)$, there are sequences $(u_{0,n})_{n\in\mathbb{N}}$ and $(u_{1,n})_{n\in\mathbb{N}}$ of $(E_{2,n}(x_n))_{n\in\mathbb{N}}$ such that u_0 is the limit of the former and u_1 is the limit of the latter. Then, for each n, the sum $u_{0,n} + u_{1,n}$ and the product $\eta\, u_{0,n}$ are in the subspace $E_{2,n}(x_n)$

and converge to $u_0 + u_1$ and $\eta\, u_0$ respectively.

(b) For each $n \in \mathbb{N}$, we have $\mathcal{H} = E_{1,n}(x_n) \oplus E_{2,n}(x_n)$; therefore, given $h \in \mathcal{H}$, there are vectors $e_{1,n} \in E_{1,n}(x_n)$ and $e_{2,n} \in E_{2,n}(x_n)$ such that $h = e_{1,n} + e_{2,n}$. As verified in the previous Lemma, the sequence $(e_{1,n})_{n\in\mathbb{N}}$ has a convergent subsequence to a vector $e \in E_1(x)$. Then the corresponding subsequence of $(e_{2,n})_{n\in\mathbb{N}}$ converges to $h - e$, which accordingly belongs to $E_2(x)$. Then $h = e + (h-e)$ is in $E_1(x) + E_2(x)$.

Moreover, if $E_1(x) \cap E_2(x) \neq \{0\}$, then there is a vector $u \neq 0$ which is the limit of a sequence $(u_{1,n})_{n\in\mathbb{N}}$ of vectors in $E_{1,n}(x_n)$ and also the limit of a sequence $(u_{2,n})_{n\in\mathbb{N}}$ inside $E_{2,n}(x_n)$. But this implies that the angle between these subspaces must be arbitrarily close to zero as n goes to ∞, which contradicts what was proved above. Therefore $\mathcal{H} = E_1(x) \oplus E_2(x)$. □

Lemma 2.8. *$E_1(x) \succ_\ell E_2(x)$.*

Proof. We must check that the ℓ-domination of the splitting at x_n is inherited by this choice of spaces at x. Fix $u \in E_2(x)$ and $v \in E_1(x) \setminus \{0\}$.

We know that v is a linear combination of $w_1, ..., w_p$ and so it is the limit of a sequence of vectors of $E_{1,n}$, say $(v_n)_{n\in\mathbb{N}}$. Therefore $A^\ell(x)v \neq 0$ because, for m big enough, the iterates $A^m(x)v$ inherit at least the minimum rate of growing of w_i, that is, $\exp(\lambda_k)$. Besides, as $\|A(x_n)v_n\| \geq \theta_\mathcal{K}$ for all n, the same inequality holds in the limit as n goes to ∞, that is, $\|A(x)v\| \geq \theta_\mathcal{K}$.

By definition of $E_2(x)$, there is a sequence in $E_{2,n}(x_n)$, say $(u_n)_{n\in\mathbb{N}}$, that converges to u. Since we have, for all n, $E_{1,n}(x_n) \succ_\ell E_{2,n}(x_n)$, taking limits on the inequality

$$\frac{\|A^\ell(x_n)u_n\|}{\|A^\ell(x_n)v_n\|} \leq \frac{1}{2},$$

we get

$$\frac{\|A^\ell(x)u\|}{\|A^\ell(x)v\|} \leq \frac{1}{2}.$$

We emphasize that, if $x \notin \mathcal{K}$, we do not known whether $E_1(x)$ and $E_2(x)$ are Oseledets-Ruelle's subspaces, since we cannot guarantee that $x \in \mathcal{O}(A)$. □

Corollary 2.9. *The subbundle $E_i(x)$, for $i = 1, 2$, is well defined and continuous.*

Proof. By definition, $E_2(x)$ does not depend on the sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{K}$ converging to x. Concerning $E_1(x)$:

(i) If $x \notin \mathcal{K}$, the ℓ-domination ensures that $E_1(x)$ is unique since its dimension is fixed: the iterates of its unit vectors grow faster than those of any unit vector in $E_2(x)$.

(ii) If $x \in \mathcal{K}$, these two subspaces coincide with the Oseledets-Ruelle's spaces already assigned to x. In fact, while constructing $E_1(x)$, we must consider the constant sequence equal to x; the Oseledets-Ruelle's subspace at x corresponding to the Lyapunov exponents λ_1, λ_2, ... λ_k is then contained in $E_1(x)$ and has the same dimension, so these two spaces coincide.

Uniqueness implies that these spaces vary continuously. □

This ends the proof of Proposition 2.5. □

2.5. **Upper semi-continuity of the entropy function.** Given a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ and a positive integer p, let $\wedge^p(\mathcal{H})$ be the p^{th} exterior power of $\mathcal{H}$, that is, the infinite dimensional space generated by p vectors of the form $e_1 \wedge e_2 \wedge ... \wedge e_p$ with $e_i \in \mathcal{H}$. The operator A induces another one on this space defined by

$$\wedge^p(A)(e_1 \wedge e_2 \wedge ... \wedge e_p) = A(e_1) \wedge A(e_2) \wedge ... \wedge A(e_p).$$

The space $\wedge^p(\mathcal{H})$ has endowed the inner product such that $\|e_1 \wedge e_2 \wedge ... \wedge e_p\|$ is the p^{th}-dimensional volume of the parallelepiped spanned by $e_1, e_2, ..., e_p$. The cocycle $\wedge^p(A)$ is continuous with respect to the associated norm. For details see [9], chapter V.

Lemma 2.10. *If A is compact and integrable, then $\wedge^p(A)$ also is.*

Proof. Fix p and A and take any bounded sequence $(y_n)_n$ of elements of $\wedge^p\mathcal{H}$; we must prove that there exists a subsequence $(y_{n_k})_k$ such that $(\wedge^p(A)(y_{n_k}))_k$ converges. For each $n \in \mathbb{N}$, let $y_n = {v_1}^n \wedge ... \wedge {v_p}^n$, with ${v_j}^n \in \mathcal{H}$ for all $j = 1, ..., p$. Hence $\wedge^p(A)(y_n) = A({v_1}^n) \wedge ... \wedge A(v_p^n)$. As $(y_n)_n$ is bounded, for each $j = 1, ..., p$ the sequence $({v_i}^n)_n$ is bounded in $\mathcal{H}$. Therefore, since A is compact, for all $j = 1, ..., p$ the sequence $A(v_j^n)_n$ admits a subsequence convergent to $u_j \in \mathcal{H}$. That is, there are subsets of $\mathbb{N}$, say $\mathbb{N}_1$, $\mathbb{N}_2$, ..., $\mathbb{N}_p$ such that

(1) $\mathbb{N}_j \supseteq \mathbb{N}_{j+1}$ for all j
(2) $A(v_j^n)_{n\in\mathbb{N}_j}$ converges to u_j.

Therefore the subsequence $(y_n)_{n\in\mathbb{N}_p}$ converges to $u_1 \wedge ... \wedge u_p$.

According to the definition of the inner product in $\wedge^p(\mathcal{H})$, for all x we have

$$\| \wedge^p (A)(x)\| \leq \|A(x)\|^p$$

and so, as A is integrable,

$$\begin{aligned} \int_X \log^+ \| \wedge^p (A)(x)\| \, d\mu(x) &\leq \int_X \log^+ \|A(x)\|^p \, d\mu(x) \\ &= p \int_X \log^+ \|A(x)\| \, d\mu(x) < \infty. \end{aligned}$$

□

Since $\wedge^p(A)$ is compact and integrable we can apply to it Theorem 2.2 and conclude that, for μ-a.e. x,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|(\wedge^p A)^n(x)\| = \lambda_1^{\wedge^p}(x).$$

This is the largest Lyapunov exponent given by the dynamics of the operator $\wedge^p(A)$ at x. Moreover, for μ-a.e. x, we have

$$\lambda_1^{\wedge^p}(x) = \sum_{i=1}^{p} \lambda_i(x)$$

and

$$\lambda_1^{\wedge^p}(x) = \lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n(x))\|$$

(see [1]). In fact, as μ is ergodic, this equality reduces, μ-a.e. x, to

$$\lambda_1^{\wedge^p} = \sum_{i=1}^{p} \lambda_i = \lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n(x))\|.$$

Given $p \in \mathbb{N}$ define the p^{th}-*entropy function* by

$$\begin{array}{rccc} LE_p : & C_I^0(X, \mathcal{C}(\mathcal{H})) & \longrightarrow & \mathbb{R} \cup \{-\infty\} \\ & A & \longmapsto & \sum_{i=1}^{p} \lambda_i(A). \end{array}$$

As the Lyapunov exponents vary in a measurable way, there is no reason to expect the function LE_p to be continuous. However, as μ is ergodic and positive on non-empty open sets, this function is upper semi-continuous. Let us see why.

Proposition 2.11. *Consider a cocycle A and the sequence given, for each $n \in \mathbb{N}$, by $a_n = \log \| \wedge^p (A^n)\|$. Then*

(i) $\lambda_1^{\wedge^p} = \mathop{lim}\limits_{n\to+\infty} \frac{a_n}{n}$.

(2i) $(a_n)_{n\in\mathbb{N}}$ *is sub-additive.*

(3i) $\mathop{lim}\limits_{n\to+\infty} \frac{a_n}{n} = \mathop{inf}\limits_{n\in\mathbb{N}} \frac{a_n}{n}$.

(4i) *For each $n \in \mathbb{N}$, the map $A \longrightarrow a_n(A)$ is continuous.*

Proof. Concerning (i):

Case 1: $\lambda_1^{\wedge^p} \geq 0$

We know that, for μ - a.e. x,

$$\lambda_1^{\wedge^p} = \lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n(x))\|,$$

so, for each such a x and n big enough, the norm $\| \wedge^p (A^n(x))\|$ is approximately $e^{\lambda_1^{\wedge^p} n}$, and therefore does not vanish. Besides, by definition,

$$\frac{1}{n} \log \| \wedge^p (A^n)\| = \frac{1}{n} \log \| \wedge^p (A^n(t_n))\|$$

for a suitable choice of $t_n \in X$. As μ is positive on non-empty open sets, the subset $\mathcal{O}(\wedge^p(A))$ is dense in X and so, for each n, there exists $z_n \in \mathcal{O}(\wedge^p(A))$ such that the distance between t_n and z_n is sufficiently small in order to guarantee that

$$\| \wedge^p (A^n(t_n)) - \wedge^p(A^n(z_n))\| \approx 0.$$

Besides, as for all n we have

$$\lim_{k\to+\infty} \frac{1}{k} \log \| \wedge^p (A^k)(z_n)\| = \lambda_1^{\wedge^p}$$

by a diagonal argument we deduce that

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n)(z_n)\| = \lambda_1^{\wedge^p}.$$

And so, if n is big enough, $\| \wedge^p (A^n(z_n))\| \geq e^{\lambda_1^{\wedge^p}} \geq 1$, and therefore

$$\frac{\| \wedge^p (A^n(t_n))\|}{\| \wedge^p (A^n(z_n))\|} \approx 1.$$

Then

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n)(t_n)\| = \lim_{n\to+\infty} \frac{1}{n} \log \frac{\| \wedge^p (A^n(t_n))\|}{\| \wedge^p (A^n(z_n))\|} . \| \wedge^p (A^n(z_n))\|$$

which reduces to

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n(t_n))\| = \lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n(z_n))\| = \lambda_1^{\wedge^p},$$

and means that

$$\lambda_1^{\wedge^p} = \lim_{n\to+\infty} \frac{a_n}{n}.$$

Case 2: $-\infty < \lambda_1^{\wedge^p} < 0$

As in the previous case, for all n and a suitable choice of t_n, we have

$$\frac{1}{n} \log \| \wedge^p (A^n)\| = \frac{1}{n} \log \| \wedge^p (A^n(t_n))\|;$$

moreover, for each n, there exists $z_n \in \mathcal{O}(\wedge^p(A))$ such that the distance between t_n and z_n is sufficiently small in order to guarantee that

$$\| \wedge^p (A^n(t_n)) - \wedge^p(A^n(z_n))\| \leq e^{\lambda_1^{\wedge^p} n}.$$

Besides, as $\lambda_1^{\wedge^p} < 0$, if n is big enough,

$$\| \wedge^p (A^n)(z_n)\| < 1.$$

Therefore

$$\frac{1}{n} \log \|\wedge^p(A^n(t_n))\| \leq \frac{1}{n} \log \left(\| \wedge^p (A^n)(z_n)\| + e^{\lambda_1^{\wedge^p} n}\right) \leq \frac{1}{n} \log \left(1 + e^{\lambda_1^{\wedge^p} n}\right)$$

and so

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n)(t_n)\| = \lambda_1^{\wedge^p}.$$

Case 3: $\lambda_1^{\wedge^p} = -\infty$

Given $\epsilon > 0$, consider, as above, sequences $(t_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that

$$\frac{1}{n} \log \| \wedge^p (A^n)\| = \frac{1}{n} \log \| \wedge^p (A^n(t_n))\|$$

and, if n is big enough,

$$\| \wedge^p (A^n(t_n)) - \wedge^p(A^n(z_n))\| \leq e^{-\epsilon n}$$

and

$$\| \wedge^p (A^n(z_n))\| \leq e^{-\epsilon n}.$$

Then

$$\frac{1}{n} \log \| \wedge^p (A^n(t_n))\| \leq \frac{1}{n} \log \left(2e^{-\epsilon n}\right)$$

and therefore, for all $\epsilon > 0$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n)(t_n)\| \leq -\epsilon$$

which implies that

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p (A^n)(t_n)\| = -\infty.$$

(2i) This assertion is the same as the one in Section 2.1.3 of (see [3]) since the extra value $-\infty$ the sequence may take does not bring any additional difficulty.

(3i) This is a direct consequence of (2i).

(4i) For each fixed n, the continuity of the map $A \longmapsto a_n(A)$ is ensured by the continuity, with the operator, of the norm $\|\wedge^p(A^n)\|$. □

Corollary 2.12. *For all p the function LE_p is upper semi-continuous.*

Proof. LE_p is the infimum of a sequence of continuous functions with values on the extended real line, and so it is upper semi-continuous (see [7]). □

3. Perturbation lemmas

Let us see how, using the absence of domination, to perform appropriate C^0-perturbations of our original system to increase the number of contractive directions.

Lemma 3.1. *Let $A \in C^0_I(X, \mathcal{C}(\mathcal{H}))$, $x \in X$ and $\epsilon > 0$. For any 2-dimensional subspace $E \subset \mathcal{H}$, we may find $\xi_0 > 0$ (not depending on x) such that, for all $\xi \in]0, \xi_0[$ there exists a measurable integrable cocycle B_ξ such that,*

(a) $B_\xi(x) \cdot u = A(x) \cdot u$, $\forall u \in E^\perp$;
(b) $B_\xi(x) \cdot u = A(x) \cdot R_\xi \cdot u$, $\forall u \in E$, *where R_ξ is the rotation of angle ξ in E;*
(c) $\|A - B_\xi\| \leq \epsilon$.

Proof. If $A = 0$, choose $B_\xi = A$. Otherwise, consider the direct sum $\mathcal{H} = E^\perp \oplus E$ and denote by

$$R_\theta = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

the matrix of the rotation of angle θ in an orthonormal basis of E. Then take $\xi_0 > 0$ such that $\|Id - R_{\xi_0}\| \leq \frac{\epsilon}{\|A\|}$ and, for each $v = v_1 + v_2$, where $v_1 \in E^\perp$ and $v_2 \in E$, define the perturbation cocycle by

$$B_\xi(y) \cdot v = \left\{ \begin{array}{ll} A(y) \cdot v & \text{if } y \neq x \\ A(x) \cdot v_1 + A(x) \cdot R_\xi \cdot v_2 & \text{if } y = x \end{array} \right\}.$$

Clearly this cocycle verifies the properties (a), (b) and (c) and the lemma is proved. $\square$

The following result tells how to interchange directions. The main idea, coming from Proposition 7.1 of [3], is to use the absence of hyperbolic behavior to concatenate several small rotations of the form given by Lemma 3.1.

Proposition 3.2. *Consider a cocycle A, $\delta > 0$ and $x \in X$ a non-periodic point endowed with a splitting $\mathcal{H} = E_1(x) \oplus E_2(x)$ such that the restriction of $A(x)$ to $E_1(x)$ is invertible and, for some $m(\delta, A) = m \in \mathbb{N}$ large enough, we have*

$$\frac{\|A^m(x)|_{E_2}\|}{\|(A^{-1}|_{E_1})^m(x)\|} \geq 1/2.$$

Then, for each $j = 0, ..., m-1$, there exists an integrable compact operator

$$L_j : \mathcal{H} \to \mathcal{H},$$

with $\|L_j - A(f^j(x))\| < \delta$ and such that $L_{m-1} \circ ... \circ L_0(v) = w$ for some nonzero vectors $v \in E_1$ and $w \in A^m(x)(E_2)$.

We now want to apply this strategy of perturbation to the set of points where domination fails.

Definition 3.1. *Let $\Lambda_p(A, m)$ be the set of points x such that, along the orbit of x we have an Oseledets-Ruelle's decomposition of index p which is m-dominated. Denote by $\Gamma_p(A, m) = X \setminus \Lambda_p(A, m)$ and by $\Gamma_p^*(A, m)$ the set of points in $\mathcal{O}(A) \cap \Gamma_p(A, m)$ which are non-periodic and verify $\lambda_p > \lambda_{p+1}$.*

As checked previously, the set $\Lambda_p(A, m)$ is closed and so all the just mentioned subsets are measurable. Notice also that if x belongs to $\Gamma_p^*(A, m)$ for some m, then the m-domination on $\mathcal{K} = \{\text{orbit of } x\}$ of the Oseledets-Ruelle's splitting may fail by two (possibly coexisting) events:

(NB) The norm of the operator A restricted to E_1 takes values arbitrarily small along the orbit of x.

That is, for all $\theta > 0$ there are $N = N_{\theta,x} \in \mathbb{N}$ and a unit vector $v_N \in E_1(f^N(x))$ such that

$$\|A(f^N(x))(v_N)\| < \theta.$$

We call $\Gamma_{p,1}^*$ the set of points $x \in \Gamma_p^*(A, m)$ where this happens.

(ND) The dynamics on the subspace E_1 does not m-dominate the one on E_2.

This means that there are $n \in \mathbb{N}$ and unit vectors $v_n \in E_1(f^n(x))$ and $u_n \in E_2(f^n(x))$ such that

$$\frac{\|A^m(f^n(x))u_n\|}{\|A^m(f^n(x))v_n\|} \geq \frac{1}{2}.$$

The points $x \in \Gamma_p^*(A, m)$ where property (ND) is valid but not (NB) will be denoted by $\Gamma_{p,2}^*$.

We proceed explaining how we can perform locally and globally a blending of specific directions of the Oseledets-Ruelle's splitting for points inside $\Gamma_{p,1}^* \cup \Gamma_{p,2}^*$. Given a point x in $\Gamma_{p,1}^* \cup \Gamma_{p,2}^*$, the aim of next lemmas is to perturb locally A, along the orbit of x, in order to carry out an abrupt decay of the norm of the p^{th}-exterior power product $\wedge^p$.

3.1. **Perturbation (p, NB).** Consider a point x in $\Gamma_{p,1}^*$. To reduce the norm of $\wedge^p$ we will just replace A by the null operator at the n^{th}-iterate of x along the direction inside $E_1(x)$ restricted to which the norm of A is very close to zero.

Lemma 3.3. *Given $\epsilon > 0$, there exists a measurable function $\mathcal{N} : \Gamma_{p,1}^* \to \mathbb{N}$ such that, for μ-almost every $x \in \Gamma_{p,1}^*$, every $n \geq \mathcal{N}(x)$ and each $j = 0, ..., n$, there exists an integrable compact operator $L_j : \mathcal{H} \to \mathcal{H}$ satisfying*

$$\|L_j - A(f^j(x))\| < \epsilon$$

and

$$\| \wedge^p (L_{n-1} \circ ... \circ L_0)\| = 0.$$

Proof. We may assume that $\mu(\Gamma^*_{p,1}) > 0$, otherwise there is nothing to prove. Therefore $\mu\left(\mathcal{O}(A) \cap \Gamma^*_{p,1}\right) = \mu\left(\Gamma^*_{p,1}\right)$. If $x \in \mathcal{O}(A) \cap \Gamma^*_{p,1}$ we can take $N_{\epsilon,x}$ as in [NB], define $\mathcal{N}(x) = N_{\epsilon,x} + 1$ and choose a unit vector $v_{\mathcal{N}} \in E_1(f^{\mathcal{N}}(x))$ such that

$$\|A(f^{\mathcal{N}}(x))(v_{\mathcal{N}})\| < \epsilon.$$

Then, for $n \geq \mathcal{N}$, consider

$$L_i = A(f^i(x)) \text{ for } i = 0, ..., \mathcal{N}-1, \mathcal{N}+1, ..., n$$

and

$$L_{\mathcal{N}}(v) = \begin{cases} A(f^{\mathcal{N}}(x))(v) & \text{if } v \in <v_{\mathcal{N}}>^{\perp} \\ 0 & \text{if } v \text{ is colinear with } v_{\mathcal{N}} \end{cases}$$

Since x is not periodic by f, this family of operators is well defined. Besides $\| \wedge^p (L_{n-1} \circ ... \circ L_0)\| = 0$. Notice also that, if ϵ is small, the above perturbation of A is also minute. □

3.2. **Perturbation (p, ND).** Consider now a point x in $\Gamma^*_{p,2}$. If among the $p+1$ first Lyapunov exponents of A the value $-\infty$ is not present, we can use the argument in [3] to alter the norm of $\wedge^p$. In the case $\lambda_{p+1} = -\infty$ we may take advantage of the fact that, in the subbundle E associated to the Lyapunov exponents λ_j for $j \geq p+1$, the norm $\|A^n(x)|_E\|$ is close to zero for n large enough.

Lemma 3.4. *Consider $\epsilon, \delta > 0$. If $m \in \mathbb{N}$ is large enough, then there exists a measurable function $\mathcal{N} : \Gamma^*_{p,2} \to \mathbb{N}$ such that, for μ-almost every $x \in \Gamma^*_{p,2}$ and every $n \geq \mathcal{N}(x)$, we may find integrable compact operators $L_0, ..., L_{n-1}$ with $\|L_j - A(f^j(x))\| < \delta$ for each j and satisfying*

(a) $\| \wedge^p (L_{n-1} \circ ... \circ L_0)\| \leq e^{n(\lambda_1 + ... + \lambda_{p-1} + \frac{\lambda_p + \lambda_{p+1}}{2} + \epsilon)}$ *if $\lambda_{p+1} \neq -\infty$;*
(b) $\| \wedge^p (L_{n-1} \circ ... \circ L_0)\| \leq e^{-n\epsilon}$ *if $\lambda_{p+1} = -\infty$.*

Proof. The proof of (a) is a direct consequence of Proposition 3.2 following the argument of Proposition 4.2 of [3].

Let us now explain the scheme to prove (b). Consider $\Gamma^*_{p,2}$, subset of $\Gamma^*_p(A, m)$ where m is large enough as demanded in Proposition 3.2. We may assume that $\mu(\Gamma^*_{p,2}) > 0$, otherwise there is nothing to prove. By definition of $\Gamma^*_p(A, m)$, we have $\lambda_p \neq -\infty$. Thus, by Lemma 3.12 of [2], we conclude that, for μ-a.e. $x \in \Gamma^*_{p,2}$, there exists $\mathcal{N}_1(x)$ such that, for all $n \geq \mathcal{N}_1(x)$ and $s \approx n/2$, the iterate $y = f^s(x)$ verifies

$$\frac{\|A^m(y)|_{E_2}\|}{\|(A^{-1}|_{E_1})^m(y)\|} \geq 1/2.$$

We can then apply Proposition 3.2 to such a generic $x \in \Gamma^*_{p,2}$ because, by hypothesis, x is not periodic and $\mathcal{H} = E_{1,x} \oplus E_{2,x}$, where $E_{1,x}$ has dimension p and corresponds to the finite vector space spanned by the Lyapunov exponents λ_1,..., λ_p (which may be not all distinct but whose multiplicities add up to p) and $E_{2,x}$ is associated to the infinite dimensional vector space spanned by the remaining ones. Therefore,

we consider, for $i = 0, ..., s, s+m, ..., n$, the operators $L_i = A(f^i(x))$ and, for the iterates $f^i(y)$ with $i = 0, ..., m-1$, we take L_i as given by Proposition 3.2.

We need now to evaluate the norm of $\wedge^p(L_{n-1} \circ ... \circ L_0)$. Take U_x the subspace associated to the largest Lyapunov exponent of the p^{th}-exterior product, say $\lambda_1^{\wedge p}$, which is given by $\lambda_1^{\wedge p} = \sum_{i=1}^p \lambda_i$. The space U_x is 1-dimensional because $\lambda_p > \lambda_{p+1}$. Denote by S_x the vector space related to the remaining Lyapunov exponents, which sum up to $\lambda_2^{\wedge p}$. To the splitting $\wedge^p(\mathcal{H}) = U \oplus S$ we may apply Lemma 4.4 of [3] and Proposition 3.2 to deduce that

$$\wedge^p(L_{m-1} \circ ... \circ L_0)(y) : \wedge^p(\mathcal{H}_y) \to \wedge^p(\mathcal{H}_{f^m(y)})$$

satisfies

$$\wedge^p (L_{m-1} \circ ... \circ L_0)(y)(U_y) \subset S_{f^m(y)}. \tag{1}$$

If A_1 denotes the action of $\wedge^p(A)$ between x and y and A_2 denotes the action of $\wedge^p(A)$ between $f^{s+m}(y)$ and $f^n(x)$, we can consider a suitable (Oseledets-Ruelle's) basis with respect to which A_1, A_2 and $B := L_{m-1} \circ ... \circ L_0(y)$ are written as the simple 4-block "matrices"

$$A_1 = \begin{pmatrix} A_1^{uu} & 0 \\ 0 & A_1^{ss} \end{pmatrix}, \; B = \begin{pmatrix} B^{uu} & B^{us} \\ B^{su} & B^{ss} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} A_2^{uu} & 0 \\ 0 & A_2^{ss} \end{pmatrix}$$

where, for $i = 1, 2$, $A_i^{uu} \in \mathbb{R}$ and A_i^{ss} is an infinite dimensional operator. It follows from (1) that $B^{uu} = 0$ and so

$$\wedge^p (L_{n-1} \circ ... \circ L_0) = \begin{pmatrix} 0 & A_2^{uu} B^{us} A_1^{ss} \\ A_2^{ss} B^{su} A_1^{uu} & A_2^{ss} B^{ss} A_1^{ss} \end{pmatrix}. \tag{2}$$

Since $\lambda_{p+1} = -\infty$, we have $\lambda_2^{\wedge p} = -\infty$ and so the operator A_i^{ss} $(i = 1, 2)$ is arbitrarily close to the null one for large choices of n. Moreover, all entries A_1^{uu}, A_2^{uu}, B^{us}, B^{su} and B^{ss} are bounded. Then it suffices to consider a large n, bigger than $\mathcal{N}_1(x)$ and m, to reach inequality (b). □

Doing these perturbations at μ-a.e. point of $\Gamma_p^*(A, m)$ we deduce that

Corollary 3.5. *Let A be a cocycle in $C_I^0(X, \mathcal{C}(\mathcal{H}))$, $\epsilon > 0$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$, $p \in \mathbb{N}$ and a cocycle $B \in C_I^0(X, \mathcal{C}(\mathcal{H}))$ with $\|B - A\|_\infty < \delta$, equal to A outside the open set $\Gamma_p(A, m)$ and verifying*

$$\lambda_1^{\wedge p}(B) < \begin{cases} [\lambda_1^{\wedge p-1}(A) + \frac{\lambda_p(A) + \lambda_{p+1}(A)}{2}] + \epsilon & \text{if } \lambda_{p+1}(A) \neq -\infty \\ -\epsilon & \text{if } \lambda_{p+1}(A) = -\infty \end{cases}.$$

Proof. After the previous lemmas, we may follow the argument in Proposition 7.3 of [3]. □

4. Proof of Theorem 1.1

Consider the p^{th}-*entropy function* defined by

$$\begin{array}{rccc} LE_p: & C_I^0(X,\mathcal{C}(\mathcal{H})) & \longrightarrow & \mathbb{R}\cup\{-\infty\} \\ & A & \longmapsto & \sum_{i=1}^{p}\lambda_i(A) \end{array}$$

where $(\lambda_i(A))_{i=1,\ldots,\infty}$ are the Lyapunov exponents of the operator $A(x)$, for every x in $\mathcal{O}(A)$. This map is upper semi-continuous, so it has a residual subset of points of continuity in the Baire set $C_I^0(X,\mathcal{C}(\mathcal{H}))$. Take A in this generic subset, consider x in the Oseledets-Ruelle's domain $\mathcal{O}(A)$ and denote by $\mathcal{K}$ the orbit of x.

If the Lyapunov exponents of $A(x)$ are all equal, then the proof is complete. Otherwise, if $p \in \mathbb{N}$ is such that $\lambda_p > \lambda_{p+1}$, we pursue as follows:

(1) If $\lambda_{p+1} > -\infty$ and x is periodic by f, say of period R, then along the orbit of x the Oseledets-Ruelle's splitting given by $E_1 \oplus E_2$, where E_1 is the subspace associated to the Lyapunov exponents $\lambda_1, \lambda_2, ..., \lambda_p$ and E_2 corresponds to the remaining ones, is m-dominated for a $m = m(x)$ large enough. In fact we have:
   - There exists $N \in \mathbb{N}$ such that $e^{N(\lambda_{p+1}-\lambda_p)} < \frac{1}{2}$.
   - For each $i \in \{1, ..., p+1\}$, there is $K_x^i \in \mathbb{N}$ such that, for all unit vector u of U_i and all positive integer $n \geq K_x^i$,
   $$\frac{1}{n}\log\|A^n(x)u\| \approx \lambda_i.$$
   - If $K(x) = \max\{N, K_x^i\}$, then, for all $n \geq K(x)$, we may conclude that
   $$\frac{\|A^{K(x)}(x)(u)\|}{\|A^{K(x)}(x)(v)\|} \leq \left(e^{N(\lambda_{p+1}-\lambda_p)}\right) < 1/2$$
   for all $u \in E_2$ and $v \in E_1$.
   - If $m = \max\{K_{f^j(x)} : j = 0, ..., R-1\}$, then along the orbit of x the Oseledets-Ruelle's splitting is m dominated.

(2) If $\lambda_{p+1} = -\infty$ and x is periodic by f, say of period R, consider, as before, the Oseledets-Ruelle's splitting given by $E_1 \oplus E_2$, where E_1 is the subspace associated to the Lyapunov exponents $\lambda_1, \lambda_2, ..., \lambda_p$ and E_2 corresponds to the remaining ones. Then:
   - Since $\lambda_p > -\infty$, there exists $\epsilon > 0$ such that $\lambda_p > -\epsilon$.
   - Therefore there exists $N \in \mathbb{N}$ such that $e^{N(-\lambda_p-\epsilon)} < \frac{1}{2}$.
   - For each $i \in \{1, ..., p\}$, there is $K_x^i \in \mathbb{N}$ such that, for all unit vector u of U_i and all positive integer $n \geq K_x^i$,
   $$\frac{1}{n}\log\|A^n(x)u\| \approx \lambda_i.$$

- There exists $N_1 \in \mathbb{N}$ such that for all unit vector u of U_{p+1} and all positive integer $n \geq N_1$,
$$\frac{1}{n} \log \|A^n(x)u\| \approx -\epsilon.$$
- If $K(x) = \max\{N, N_1, K_x^i\}$, then, for all $n \geq K(x)$, we may conclude that
$$\frac{\|A^{K(x)}(x)(u)\|}{\|A^{K(x)}(x)(v)\|} \leq \left(e^{N(-\lambda_p - \epsilon)}\right) < 1/2$$
for all $u \in E_2$ and $v \in E_1$.
- If $m = \max\{K_{f^j(x)} : j = 0, ..., R-1\}$, then along the orbit of x the Oseledets-Ruelle's splitting is m dominated.

(3) If x is non-periodic and the Oseledets-Ruelle's splitting along the orbit of x is m-dominated for some m, the proof ends.

(4) Finally if x is non-periodic, belongs to $\Gamma_p^*(A, m)$ for all m and one of these subsets, say $\Gamma_p^*(A, m_0)$, has positive μ measure, then the m_0-domination on $\mathcal{K}$ of the Oseledets-Ruelle's splitting may fail because x is in one of the corresponding sets $\Gamma_{p,1}^*$ or $\Gamma_{p,2}^*$. As seen in the previous corollary, given ϵ, in both cases there is a cocycle $B \in C_I^0(X, \mathcal{C}(\mathcal{H}))$ such that $\|A - B\|$ is arbitrarily small but
- $|LE_p(A) - LE_p(B)| > \epsilon$, in the case $\lambda_{p+1}(A) \neq -\infty$
- $LE_p(B) = -\infty$ while $LE_p(A)$ is finite, when $\lambda_{p+1}(A) = -\infty$

which contradicts the continuity at A of the map LE_p.

Acknowledgements

MB was Supported by Fundação para a Ciência e a Tecnologia, SFRH/BPD/20890/2004. MC was Partially supported by Fundação para a Ciência e a Tecnologia through CMUP.

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